THE HOMOLOGY OF THE REAL PROJECTIVE PLANE

Group 2

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ROAD MAP

Definitions

Two definitions of the Real Projective Plane

Computation

Derive the Betti Numbers for the first three homology groups

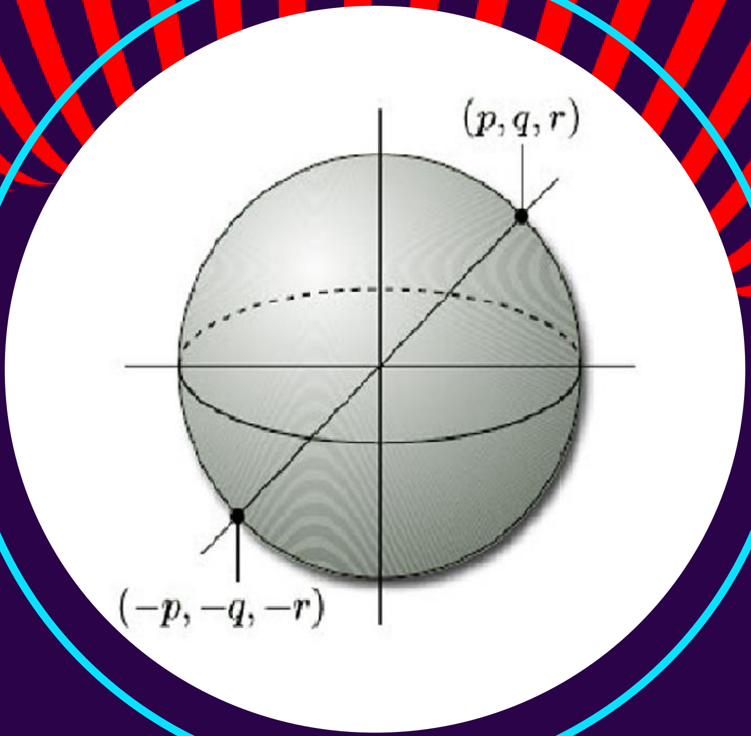
Parametrization and R Studio Code

Find a Parametrization and introduce the code used to analyze the Real Projective Plane

DEFINITION OF THE REAL PROJECTIVE PLANE

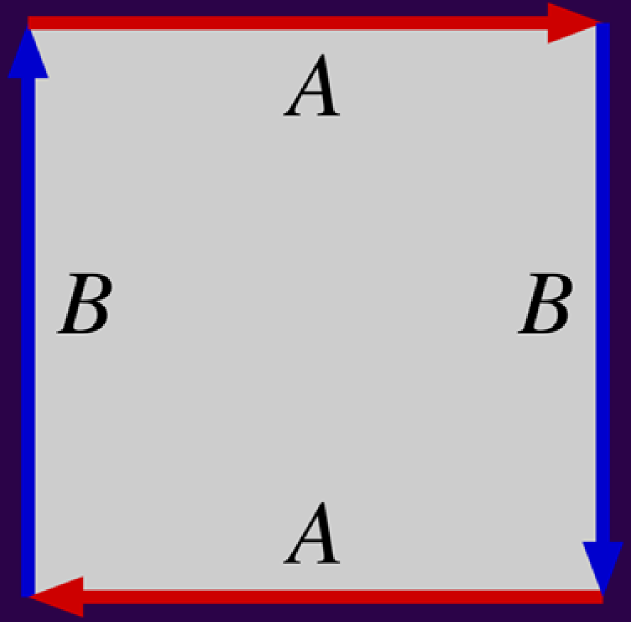
The real projective plane is a topological space that has many applications. The most common representation of the real projective plane is the collection of lines that pass through the origin in $\mathbb{R}^3$. This means that each line that passes through the origin and all the points that lie on that line are now considered as one "point." The unit sphere (S^2) can be used as a restriction to formulate an example of the real projective plane. Notice that if we group up a point on the unit sphere with its antipodal point, we can create a line that passes through the origin of the unit sphere. Thus, we can now define the real projective plane as the Quotient Space:

$$S^2/\sim \qquad (x \sim y \; iff \; x = \lambda y, \lambda \neq 0)$$



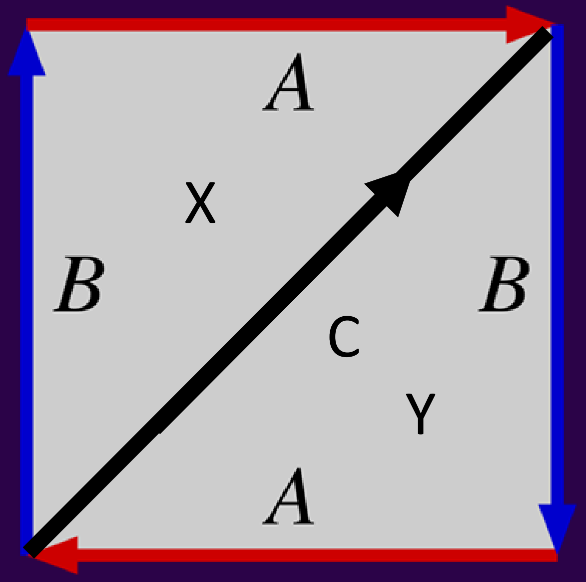
THE FUNDAMENTAL POLYGON OF THE REAL PROJECTIVE SPACE

The Δ-Complex of the Real Projective Space can be seen below. The derivation of this polygon is born from the representation of the Mobius Strip. If we use the single-edge polygon representation of the mobius strip and "glue" the opposite open edges together, we obtain the real projective plane. Note that the polygon contains two vertices and two edges.



COMPUTING THE BETTI VALUES OF THE REAL PROJECTIVE PLANE (IN $\mathbb{Z}_2$)

In order to compute the Betti numbers for the Real Projective Space, It is important to triangulate the polygon. This lets us use simplicial complexes to make the derivation easier. Note that since our Betti numbers will change depending on the field used, $\mathbb{Z}_2$ will be used since the TDA Package in R uses $\mathbb{Z}_2$ to form the Persistent Homology and Persistent Barcode. It is easy to triangulate the polygon representation by adding a diagonal edge to the polygon.



BETTI NUMBERS

	$\mathbf{RP^2}$
β_0	1
β_1	$\begin{cases} 1\ if\ F = \mathbb{Z}_2 \\ 0\ otherwise \end{cases}$
β_2	$\begin{cases} 1\ if\ F = \mathbb{Z}_2 \\ 0\ otherwise \end{cases}$

A PARAMETRIZATION OF RP^2 EMBEDDED IN R^4

Define

$$f: S^2 \to \mathbb{R}^4, \qquad (x, y, z) \to (xy, xz, y^2 - z^2, 2yz)$$

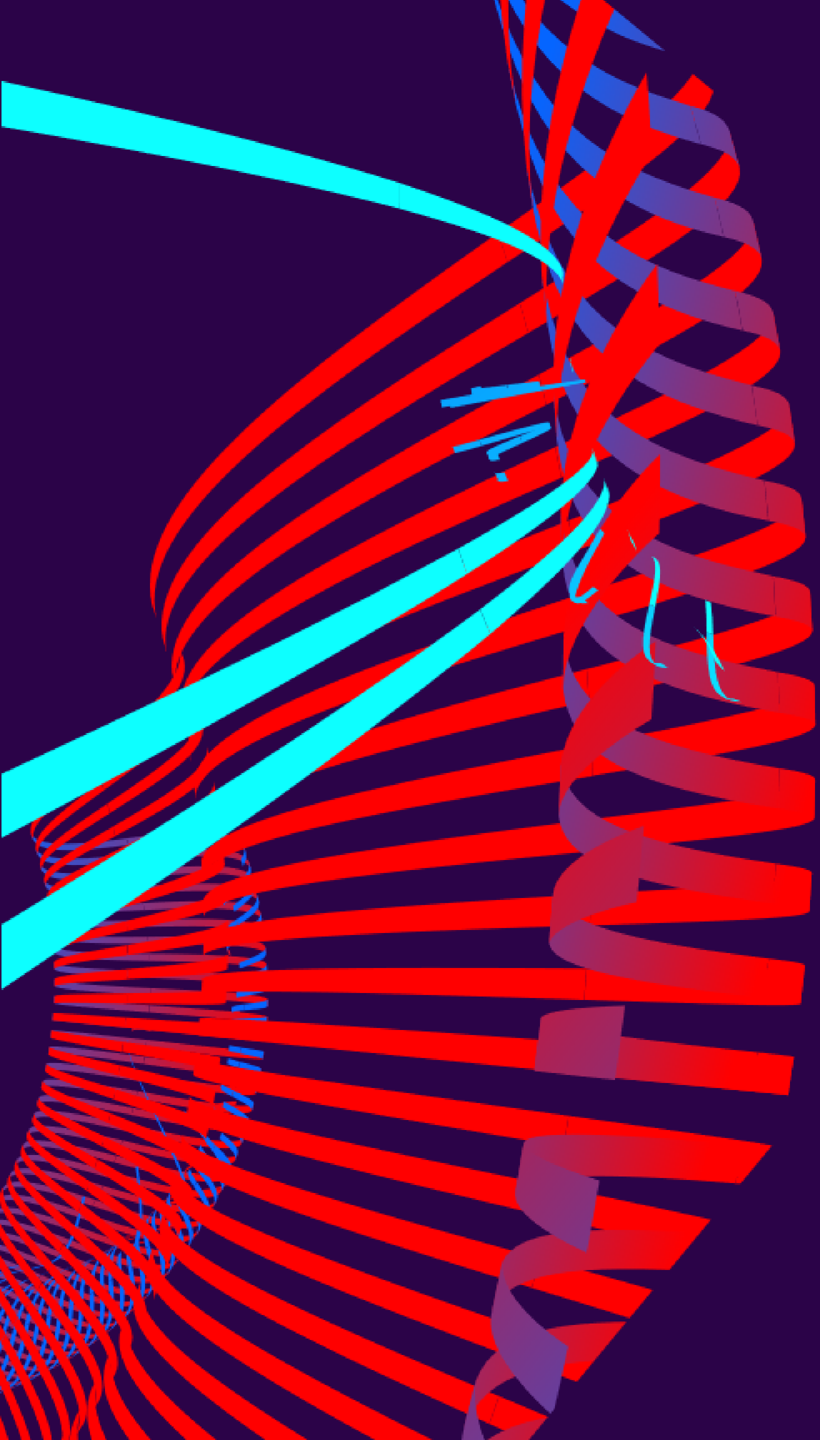
Notice that this function is specifically crafted to result in:

$$f(x, y, z) = f(x', y', z')\ if\ and\ only\ if\ (x', y', z') = \pm(x, y, z)$$

This is a requirement for the real projective plane. Then, the composition

$$f(S^2)$$

is an embedded of RP^2 in R^4. It is useful to use this composition to analyze the persistent barcode of the real projective plane using the TDA packages nested within R Studio.



THANK YOU!

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